

# A criterion for a proper rational map to be equivalent to a proper polynomial map

*Dedicated to Professor Kohn on the occasion of his 75th birthday*

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## 1 Introduction

Let  $\mathbb{B}^n$  be the unit ball in the complex space  $\mathbb{C}^n$ . Write  $Rat(\mathbb{B}^n, \mathbb{B}^N)$  for the space of proper rational holomorphic maps from  $\mathbb{B}^n$  into  $\mathbb{B}^N$  and  $Poly(\mathbb{B}^n, \mathbb{B}^N)$  for the set of proper polynomial holomorphic maps from  $\mathbb{B}^n$  into  $\mathbb{B}^N$ . We say that  $F$  and  $G \in Rat(\mathbb{B}^n, \mathbb{B}^N)$  are *equivalent* if there are automorphisms  $\sigma \in Aut(\mathbb{B}^n)$  and  $\tau \in Aut(\mathbb{B}^N)$  such that  $F = \tau \circ G \circ \sigma$ .

Proper rational holomorphic maps from  $\mathbb{B}^n$  into  $\mathbb{B}^N$  with  $N \leq 2n - 2$  are equivalent to the identity map ([Fa] [Hu]). In [HJX], it is shown that  $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$  with  $N \leq 3n - 4$  is equivalent to a quadratic monomial map, called the D'Angelo map. However, when the codimension is sufficiently large, there are a plenty of rooms to construct rational holomorphic maps with certain arbitrariness by the work in Catlin-D'Angelo [CD]. Hence, it is reasonable to believe that after lifting the codimension restriction, many proper rational holomorphic maps are not equivalent to polynomial proper holomorphic maps. In the last paragraph of the paper [DA], D'Angelo gave a philosophic discussion on this matter. However, explicit examples of proper rational holomorphic maps, that are not equivalent to polynomial proper holomorphic maps, do not seem to exist in the literature. And the problem of determining if an explicit proper rational holomorphic map is equivalent to a polynomial holomorphic map does not seem to have been studied so far.

This short paper is concerned with such a problem. We will first give an explicit criterion when a rational holomorphic map is equivalent to a polynomial holomorphic map. Making

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use of this criterion, we construct rational holomorphic maps of degree 3 that are ‘almost’ linear but are not equivalent to polynomial holomorphic maps. On the other hand, with the help of the classification result in [CJX], our criterion is used in this paper to show that any proper rational holomorphic map from  $\mathbb{B}^2$  into  $\mathbb{B}^N$  of degree two is equivalent to a polynomial holomorphic map.

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## 2 A criterion

Let  $F = \frac{P}{q} = \frac{(P_1, \dots, P_N)}{q}$  be a rational holomorphic map from the unit ball  $\mathbb{B}^n \subset \mathbb{C}^n$  into the unit ball  $\mathbb{B}^N \subset \mathbb{C}^N$ , where  $(P_j)_{j=1}^N, q$  are polynomial holomorphic functions and  $(P_1, \dots, P_N, q) = 1$ . We define  $\deg(F) = \max\{\deg(P_j)_{j=1}^N, \deg(q)\}$ . Then  $F$  induces a rational map from  $\mathbb{CP}^n$  into  $\mathbb{CP}^N$  given by

$$\hat{F}([z_1 : \dots : z_n : t]) = \left[ t^k P\left(\frac{z}{t}\right) : t^k q\left(\frac{z}{t}\right) \right]$$

where  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$  and  $\deg(F) = k$ .

$\hat{F}$  may not be holomorphic in general. For instance, we have the following:

### Example 2.1

I. Let  $F_\theta(z, w) = (z, \cos\theta w, \sin\theta zw, \sin\theta w^2)$  be the proper monomial map from  $\mathbb{B}^2$  into  $\mathbb{B}^4$  (called the D’Angelo map), where  $0 < \theta < \frac{\pi}{2}$ . Then the pole of  $\hat{F}_\theta$  consists of one point:  $\{[z : w : t] \mid w = 0, t = 0\} = \{[1 : 0 : 0]\}$ .

II. Let  $G_\alpha = (z^2, \sqrt{1 + \cos^2\alpha} zw, \cos\alpha w^2, \sin\alpha w)$  be the proper monomial map from  $\mathbb{B}^2$  into  $\mathbb{B}^4$  where  $0 \leq \alpha < \frac{\pi}{2}$ . Then  $G_\alpha$  induces

$$\hat{G}_\alpha = [z^2 : \sqrt{1 + \cos^2\alpha} zw : \cos\alpha w^2 : \sin\alpha wt : t^2].$$

There is no pole for  $\hat{G}_\alpha$ . Hence  $\hat{G}_\alpha$  is holomorphic.

Write  $\mathbb{B}_1^n = \{[z_1 : \dots : z_n : t] \in \mathbb{CP}^n \mid \sum_{j=1}^n |z_j|^2 < |t|^2\}$ , which is the projectivization of  $\mathbb{B}^n$ . Write  $U(n+1, 1)$  for the collection of the linear transforms  $A$  such that

$$AE_{n+1,1}\overline{A}^t = E_{n+1,1}$$

where

$$E_{n+1,1} = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}.$$

Then  $U(n+1,1)/\{\pm Id\} = Aut(\mathbb{B}_1^n) \approx Aut(\mathbb{B}^n)$ .

**Lemma 2.1** *For any hyperplane  $H \subset \mathbb{CP}^n$  with  $H \cap \overline{\mathbb{B}_1^n} = \emptyset$ , there is a  $\sigma \in U(n+1,1)$  such that  $\sigma(H) = H_\infty = \{[z_1 : \dots : z_n : 0]\}$ .*

*Proof:* Assume that  $H : \sum_{j=1}^n a_j z_j - a_{n+1} t = 0$  with  $\vec{a} = (a_1, \dots, a_{n+1}) \neq 0$ . Under the assumption that  $H \cap \overline{\mathbb{B}_1^n} = \emptyset$ , we have  $a_{n+1} \neq 0$ . Without loss of generality, we can assume  $a_{n+1} = 1$ . Let  $U$  be an  $n \times n$  unitary matrix such that

$$(a_1, \dots, a_n)U = (\lambda, 0, \dots, 0).$$

Let  $\sigma = \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix}$ . Then  $\sigma(H) = \{[z : t] \in \mathbb{CP}^n \mid \lambda z_1 - t = 0\}$  with  $|\lambda| < 1$ . Let  $T \in Aut(\mathbb{B}^n)$  be defined by

$$T(z, z') = \left( \frac{z_1 - \bar{\lambda}}{1 - \lambda z_1}, \frac{\sqrt{1 - |\lambda|^2} z'}{1 - \lambda z_1} \right)$$

with  $z' = (z_2, \dots, z_n)$ . Then  $\hat{T} \in U(n+1,1)$  is defined by

$$\hat{T}([z : t]) = [z_1 - \bar{\lambda}t : \sqrt{1 - |\lambda|^2} z' : t - \lambda z_1].$$

Then  $\hat{T} \circ \sigma$  maps  $H$  to  $H_\infty$ .  $\square$

Our criterion can be stated as follows:

**Theorem 2.2** *Let  $F \in Rat(\mathbb{B}^n, \mathbb{B}^N)$ . Then  $F$  is equivalent to a proper polynomial holomorphic map from  $\mathbb{B}^n$  to  $\mathbb{B}^N$  if and only if there exist hyperplanes  $H \subset \mathbb{CP}^n$  and  $H' \subset \mathbb{CP}^N$  such that  $H \cap \overline{\mathbb{B}_1^n} = \emptyset$ ,  $H' \cap \overline{\mathbb{B}_1^N} = \emptyset$  and*

$$\hat{F}(H) \subset H', \quad \hat{F}(\mathbb{CP}^n \setminus H) \subset \mathbb{CP}^N \setminus H'.$$

*Proof:* If  $F$  is a polynomial holomorphic map, then  $\hat{F} = [t^k F(\frac{z}{t}), t^k]$  with  $\deg(F) = k$ . Let  $H = H_\infty$  and  $H' = H'_\infty$ . Then they satisfy the property defined in the lemma.

If  $F$  is equivalent to a polynomial holomorphic map  $G$ , then there exist  $\hat{\sigma} \in U(n+1,1)$ ,  $\hat{\tau} \in U(n+1,1)$  such that  $\hat{F} = \hat{\tau} \circ \hat{G} \circ \hat{\sigma}$ . Let  $H = \hat{\sigma}^{-1}(H_\infty)$  and  $H' = \hat{\tau}(H'_\infty)$ . Then they are the desired ones.

Conversely, suppose that  $\hat{F}$ ,  $H$  and  $H'$  are as in the theorem. By Lemma 2.1, we can find  $\hat{\sigma} \in U(n+1,1)$  and  $\hat{\tau} \in U(n+1,1)$  such that  $\hat{\sigma}(H) = H_\infty$  and  $\hat{\tau}(H') = H'_\infty$ . Let

$\hat{Q} = \hat{\tau} \circ \hat{F} \circ \hat{\sigma}^{-1}$ . Then  $\hat{Q}$  induces a proper rational holomorphic map  $Q$  from  $\mathbb{B}^n$  into  $\mathbb{B}^N$ . If  $Q = \frac{P}{q}$  where  $(P, q) = 1$  and  $\deg(Q) = k$ , then

$$\hat{Q} = [t^k P(\frac{z}{t}) : t^k q(\frac{z}{t})].$$

Suppose that  $q \neq \text{constant}$ . Let  $z_0 \in \mathbb{C}^n$  be such that  $q(z_0) = 0$ . Notice that  $\hat{Q}(H_\infty) \subset H'_\infty$  and  $\hat{Q}(\mathbb{CP}^n \setminus H_\infty) \subset \mathbb{CP}^N \setminus H'_\infty$ . However  $t^k q(\frac{z_0}{t}) = 0$  for  $t = 1$ . Hence,  $\hat{Q}([z_0 : 1]) \subset H'_\infty$ . That gives a contradiction.  $\square$

Write the Cayley transformation

$$\rho_n(z', z_n) = \left( \frac{2z'}{1 - iz_n}, \frac{1 + iz_n}{1 - iz_n} \right).$$

Then  $\rho_n$  biholomorphically maps  $\partial\mathbb{H}^n$  to  $\partial\mathbb{B}^n \setminus \{(0, 1)\}$ .  $\rho_n$  induces a linear transformation of  $\mathbb{CP}^n$ :

$$\hat{\rho}_n = [2z' : t + iz_n : t - iz_n].$$

$\hat{\rho}_n$  maps  $\mathbf{S}_1^n = \{[z : t] \in \mathbb{CP}^n \mid \frac{z_n \bar{t} - t \bar{z}_n}{2i} > |z'|^2\}$  to  $\mathbb{B}_1^n$ .

Now let  $G$  be a CR map from  $\partial\mathbb{H}^n$  to  $\partial\mathbb{H}^N$ . Then  $\rho_N \circ G \circ \rho_n^{-1}$  extends to a proper rational holomorphic map from  $\mathbb{B}^n$  to  $\mathbb{B}^N$ . Then, Theorem 2.2 can be restated as:

**Theorem 2.3**  $\rho_N \circ G \circ \rho_n^{-1}$  is equivalent to a polynomial holomorphic map if and only if there are  $H \subset \mathbb{CP}^n$ ,  $H' \subset \mathbb{CP}^N$  such that  $\hat{G}(H) \subset H'$  and  $\hat{G}^{-1}(H') \subset H$  with

$$H \cap \overline{\mathbf{S}_1^n} = \emptyset, \quad H' \cap \overline{\mathbf{S}_1^N} = \emptyset.$$

### 3 proper rational holomorphic maps from $\mathbb{B}^2$ into $\mathbb{B}^N$ of degree two

As a first application of Theorem 2.2, we prove the following:

**Theorem 3.1** A map  $F \in \text{Rat}(\mathbb{B}^2, \mathbb{B}^N)$  of degree two is equivalent to a polynomial proper holomorphic map in  $\text{Poly}(\mathbb{B}^2, \mathbb{B}^N)$ .

*Proof:* By [HJX], we know that any rational holomorphic map of degree 2 from  $\mathbb{B}^2$  into  $\mathbb{B}^N$  is equivalent to a map of the form  $(G, 0)$ , where the map  $G$  is from  $\mathbb{B}^2$  into  $\mathbb{B}^5$ . Hence, to prove Theorem 3.1, we need only to assume that  $N = 5$ . After applying a Cayley transformation

and using the result in [CJX], we can assume that  $F = (f, \phi_1, \phi_2, \phi_3, g)$  is from  $\mathbb{H}^2$  into  $\mathbb{H}^5$  with either

(I)

$$f = \frac{z + \frac{i}{2}zw}{1 + e_2w^2}, \quad \phi_1 = \frac{z^2}{1 + e_2w^2}, \quad \phi_2 = \frac{c_1zw}{1 + e_2w^2}, \quad \phi_3 = 0, \quad g = \frac{w}{1 + e_2w^2}$$

where  $-e_2 = \frac{1}{4} + c_1^2$  and  $c_1 > 0$  or

(II)

$$f = \frac{z + (\frac{i}{2} + ie_1)zw}{1 + ie_1w + e_2w^2}, \quad \phi_1 = \frac{z^2}{1 + ie_1w + e_2w^2},$$

$$\phi_2 = \frac{c_1zw}{1 + ie_1w + e_2w^2}, \quad \phi_3 = \frac{c_3w^2}{1 + ie_1w + e_2w^2}, \quad g = \frac{w + ie_1w^2}{1 + ie_1w + e_2w^2}$$

where  $-e_1, -e_2 > 0$ ,  $c_1, c_3 > 0$  and

$$e_1e_2 = c_3^2, \quad -e_1 - e_2 = \frac{1}{4} + c_1^2.$$

Write  $[z : w : t]$  for the homogeneous coordinates of  $\mathbb{CP}^2$ , then in Case (I) it induces a meromorphic map  $\hat{F} : \mathbb{CP}^2 \rightarrow \mathbb{CP}^5$  given by

$$\hat{F}([z : w : t]) = [tz + \frac{i}{2}zw : z^2 : c_1zw : 0 : tw : t^2 + e_2w^2] \quad \forall [z : w : t] \in \mathbb{CP}^2,$$

and in Case (II), it induces a meromorphic map  $\hat{F} : \mathbb{CP}^2 \rightarrow \mathbb{CP}^5$  given by

$$\hat{F}([z : w : t]) = [tz + (\frac{i}{2} + ie_1)zw : z^2 : c_1zw : c_3w^2 : tw + ie_1w^2 : t^2 + ie_1wt + e_2w^2]$$

$\forall [z : w : t] \in \mathbb{CP}^2$ .

In terms of Theorem 2.3, we will look for  $H = \{-t = \mu_1z_1 + \mu_2z_2\} \subset \mathbb{CP}^2$  and  $H' = \{-t' = \sum_{j=1}^5 \lambda_j z'_j\} \subset \mathbb{CP}^5$  such that  $H \cap \overline{\mathbf{S}}_1^2 = \emptyset$ ,  $H' \cap \overline{\mathbf{S}}_1^5 = \emptyset$  with

$$\hat{F}(H) \subset H' \quad \text{and} \quad \hat{F}^{-1}(H') \subset H.$$

We next prove the following lemma:

**Lemma 3.2** *Let  $H = \{-t = \sum_{j=1}^n K_j z_j\} \subset \mathbb{CP}^n$ . Then  $H \cap \overline{\mathbf{S}}_1^n = \emptyset$  if and only if*

$$4\Im(K_n) + \sum_{j=1}^{n-1} |K_j|^2 < 0.$$

*Proof:* Suppose for  $z_j$  and  $t = -\sum_{j=1}^n K_j z_j$ , we have

$$\frac{w\bar{t} - t\bar{w}}{2i} < \sum_{j=1}^{n-1} |z_j|^2.$$

Here we identify  $z_n = w$ . We then get

$$\frac{-\overline{K_n}|w|^2 + K_n|w|^2}{2i} + \sum_{j=1}^{n-1} \frac{-\overline{K_j}z_j w + K_j z_j \bar{w}}{2i} < \sum_{j=1}^{n-1} |z_j|^2.$$

Hence

$$|w|^2 \Im(K_n) < \sum_{j=1}^{n-1} \{|z_j|^2 - 2\Re(\frac{K_j}{2i} z_j \bar{w})\},$$

or

$$|w|^2 \left( \Im(K_n) + \sum_{j=1}^{n-1} \frac{|K_j|^2}{4} \right) < \sum_{j=1}^{n-1} |z_j - \frac{\overline{K_j}}{2i} w|^2.$$

Since  $\{z_j, w\}$  are independent variables, this can only happen if and only if

$$\Im(K_n) + \sum_{j=1}^{n-1} \frac{|K_j|^2}{4} < 0.$$

This proves the lemma.  $\square$

We remark that for the map  $\hat{F}$  as defined above, if  $H$  and  $H'$  are hyperplanes sitting in  $\mathbb{CP}^2$  and  $\mathbb{CP}^5$ , respectively, such that

$$H \cap \mathbf{S}_1^2 = \emptyset, \quad H' \cap \mathbf{S}_1^5 = \emptyset, \quad \hat{F}(H) \subset H', \quad \hat{F}^{-1}(H') \subset H,$$

then  $H$  and  $H'$  have to be defined, respectively, by equations of the form:

$$H : -t = \mu_1 z + \mu_2 w, \quad H' : -t' = \sum_{j=1}^5 \lambda_j z'_j.$$

Indeed, if  $H' = \{\sum_{j=1}^5 \lambda_j z'_j = 0\}$ , then

$$\lambda_1 [tz + i(\frac{1}{2} + e_1)zw] + \lambda_2 z^2 + \lambda_3 c_1 zw + \lambda_4 c_3 w^2 + \lambda_5 (tw + ie_1 w^2) = (\mu_0 t + \mu_1 z + \mu_2 w)^2$$

$\forall [z : w : t] \in \mathbb{CP}^2$ .

We notice that  $\lambda_5 \neq 0$ ; otherwise  $H' \cap \mathbf{S}_1^5 \neq \emptyset$ . We thus get  $\mu_0 = 0$ . This would immediately give a contradiction. Similarly, if  $H = \{\mu_1 z + \mu_2 w = 0\}$ , then it follows that  $H'$  is defined by  $\{\sum_{j=1}^5 \lambda_j z'_j = 0\}$ . We also reach a contradiction.

Summarizing the above, we have:

**Case (I)** We need only to find out  $\mu_1, \mu_2, \lambda_1, \dots, \lambda_5 \in \mathbb{C}$  such that

$$4\Im(\mu_2) + |\mu_1|^2 < 0, \quad 4\Im(\lambda_5) + \sum_{j=1}^4 |\lambda_j|^2 < 0$$

and

$$\lambda_1(tz + \frac{i}{2}zw) + \lambda_2 z^2 + \lambda_3 c_1 zw + \lambda_5 tw + (t^2 + e_2 w^2) = (t + \mu_1 z + \mu_2 w)^2 \quad \forall [z : w : t] \in \mathbb{CP}^2.$$

It is easy to verify that  $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \mu_1 = 0$ ,  $\lambda_5 = -2\sqrt{|e_2|}i$  and  $\mu_2 = -\sqrt{|e_2|}i$  satisfy the above conditions. Hence in case (I), the map is always equivalent to a polynomial holomorphic map.

**Case (II)** Similar to case (I), we need to find out  $\mu_1, \mu_2, \lambda_1, \dots, \lambda_5 \in \mathbb{C}$  such that

$$4\Im(\mu_2) + |\mu_1|^2 < 0, \quad 4\Im(\lambda_5) + \sum_{j=1}^4 |\lambda_j|^2 < 0$$

and

$$\begin{aligned} & \lambda_1(tz + i(\frac{1}{2} + e_1)zw) + \lambda_2 z^2 + \lambda_3 c_1 zw + \lambda_4 c_3 w^2 + \lambda_5(tw + ie_1 w^2) \\ & + (t^2 + ie_1 tw + e_2 w^2) \equiv (t + \mu_1 z + \mu_2 w)^2, \quad \forall [z : w : t] \in \mathbb{CP}^2. \end{aligned}$$

Comparing the coefficients, we get

$$\begin{aligned} \lambda_1 &= 2\mu_1, \quad \lambda_2 = \mu_1^2, \quad \lambda_3 = \frac{1}{c_1}[-i(1 + 2e_1)\mu_1 + 2\mu_1\mu_2], \\ \lambda_4 &= \frac{1}{c_3}(\mu_2^2 - e_2 - 2ie_1\mu_2 - e_1^2), \quad \lambda_5 = 2\mu_2 - ie_1. \end{aligned}$$

In sum, we obtain the following statement:

$\rho_N \circ F \circ \rho_n^{-1}$  is equivalent to a polynomial holomorphic map if and only if there are  $\mu_1, \mu_2 \in \mathbb{C}$  such that  $4\Im(\mu_2) + |\mu_1|^2 < 0$  and that

$$-4e_1 + 8\Im(\mu_2) + 4|\mu_1|^2 + |\mu_1|^4 + \frac{1}{c_1^2}|2\mu_1\mu_2 - i(1 + 2e_1)\mu_1|^2 + \frac{1}{c_3^2}|\mu_2^2 - e_2 - e_1^2 - 2ie_1\mu_2|^2 < 0.$$

We will look for  $\mu_1$  and  $\mu_2$  with  $\mu_1 = 0$  and  $\mu_2 = iy$  ( $y < 0$ ).

To prove that  $\rho_N \circ F \circ \rho_n^{-1}$  is equivalent to a polynomial holomorphic map, it suffices for us to show that there exists  $y < 0$  such that

$$-4e_1 + 8y + \frac{1}{c_3^2}(-y^2 - e_2 - e_1^2 + 2e_1y)^2 < 0,$$

or

$$J(y) := (-4e_1 + 8y)e_1e_2 + (y^2 - 2e_1y + e_1^2 + e_2)^2 = (8y - 4e_1)e_1e_2 + ((y - e_1)^2 + e_2)^2 < 0.$$

Notice that as a function in  $y < 0$ ,

$$\lim_{y \rightarrow -\infty} J(y) = +\infty, \quad J(0) = (e_1^2 - e_2)^2 > 0.$$

We need to show that

$$\min_{y \leq 0} J(y) < 0.$$

Notice that  $J'(y) = 8e_1e_2 + 4((y - e_1)^2 + e_2)(y - e_1)$ . Setting  $J'(y) = 0$ , we get

$$(y - e_1)^3 + e_2(y - e_1) + 2e_1e_2 = 0.$$

$J'(y) = 0$  thus has a root  $y_0 \in (-\infty, 0)$ ; for

$$\lim_{y \rightarrow -\infty} J'(y) = -\infty, \quad J'(0) = 4(-e_1^3 + e_1e_2) > 0.$$

Let  $\zeta_0, \zeta_1, \zeta_2$  be the solution of

$$\zeta^3 + e_2\zeta + 2e_1e_2 = 0 \quad \text{with } \zeta_0 = y_0 - e_1.$$

Then  $\zeta_0 + \zeta_1 + \zeta_2 = 0$ ,  $\zeta_0\zeta_1 + \zeta_0\zeta_2 + \zeta_1\zeta_2 = e_2$  and  $\zeta_0\zeta_1\zeta_2 = -2e_1e_2$ . Hence  $\zeta_0 = -\zeta_1 - \zeta_2$ . We get

$$-\zeta_0^2 + \zeta_1\zeta_2 = e_2,$$



or  $\zeta_1\zeta_2 = e_2 + \zeta_0^2$ , and

$$\frac{1}{\zeta_1\zeta_2} = -\frac{\zeta_0}{2e_1e_2}.$$

In particular,  $\frac{1}{\zeta_1\zeta_2} \in \mathbb{R} \setminus \{0\}$ .

Now  $J(y_0) = (-4e_1 + 8\zeta_0 + 8e_1)e_1e_2 + (\zeta_0^2 + e_2)^2 = 2e_1e_2(4\zeta_0 + 2e_1) + (\zeta_1\zeta_2)^2 = -\zeta_1\zeta_1\zeta_2(4\zeta_0 + 2e_1) + (\zeta_1\zeta_2)^2$ .

Notice that  $4\zeta_0^3 = -8e_1e_2 - 4e_2\zeta_0$ . We see that

$$\begin{aligned} 2e_1e_2 \frac{J(y_0)}{(\zeta_1\zeta_2)^2} &= 2e_1e_2 + \zeta_0^2(4\zeta_0 + 2\zeta_1) = 2e_1e_2 - 8e_1e_2 - 4e_2\zeta_0 + 2e_1\zeta_0^2 \\ &= -6e_1e_2 - 4e_2\zeta_0 + 2e_1\zeta_0^2 = -2e_2(3e_1 + 2\zeta_0) + 2e_1\zeta_0^2. \end{aligned}$$

Since  $\zeta_0 = y_0 - e_1 < -e_1$ ,  $3e_1 + 2\zeta_0 < e_1 < 0$ . Therefore  $\frac{J(y_0)}{(\zeta_1\zeta_2)^2} 2e_1e_2 < 0$ . Here we showed that  $J(y_0) < 0$ . This completes the proof of Theorem 3.1.  $\square$

Our proof of Theorem 3.1 is, in fact, a constructive proof, which can be used to find precisely the polynomial holomorphic maps equivalent to the original ones. In the following, we demonstrate this by giving an explicit example:

**Example 3.3** Let  $F = (f, \phi_1, \phi_2, \phi_3, g) : \mathbb{H}^2 \rightarrow \mathbb{H}^5$  be defined as follows:

$$\begin{aligned} f(z, w) &= \frac{z - \frac{i}{2}zw}{1 - iw - \frac{1}{3}w^2}, \quad \phi_1(z, w) = \frac{z^2}{1 - iw - \frac{1}{3}w^2}, \\ \phi_2(z, w) &= \frac{\sqrt{\frac{13}{12}}zw}{1 - iw - \frac{1}{3}w^2}, \quad \phi_3(z, w) = \frac{\frac{\sqrt{3}}{3}w^2}{1 - iw - \frac{1}{3}w^2}, \quad g(z, w) = \frac{w - iw^2}{1 - iw - \frac{1}{3}w^2} \end{aligned}$$

It is equivalent to the proper polynomial holomorphic map  $G$  from  $\mathbb{B}^2$  into  $\mathbb{B}^5$ :

$$G(z, w) = \left( \frac{\sqrt{3}}{9}(-2 + 4z + z^2), -\frac{\sqrt{6}}{9}(1 + z + z^2), \frac{\sqrt{3}}{12}(5 + 3z)w, \frac{\sqrt{6}}{6}w^2, \frac{\sqrt{13}}{12}i(1 - z)w \right).$$

*Proof:* In fact, for the map  $F$  given above,  $e_1 = -1$ ,  $e_2 = -\frac{1}{3}$ ,  $c_1 = \sqrt{\frac{13}{12}}$ ,  $c_3 = \frac{\sqrt{3}}{3}$ . From the proof of Theorem 3.1,  $\hat{F}(H) = H'$  where  $H \subset \mathbb{CP}^2$ ,  $H' \subset \mathbb{CP}^5$  are defined by

$$\begin{aligned} H : t &= -y_0iw, \text{ or } \frac{w}{t} = \frac{i}{y_0}, \\ H' : t' &= -\lambda_4z'_4 - \lambda_5w', \text{ or } -\lambda_4\frac{z'_4}{t'} - \lambda_5\frac{w'}{t'} = 1. \end{aligned}$$

Here  $y_0 < 0$  is a solution for  $(y_0 + 1)^3 - \frac{1}{3}(y_0 + 1) + \frac{2}{3} = 0$ ,  $\lambda_4 = \frac{1}{c_3}[-(y_0 - e_1)^2 - e_2] = -\frac{(y_0 - e_1)^2 + e_2}{\sqrt{e_1 e_2}}$  and  $\lambda_5 = 2iy_0 - e_1 i$ . Therefore  $y_0 = -2$ ,  $\lambda_4 = -\frac{2}{\sqrt{3}}$  and  $\lambda_5 = -3i$ . Thus we see that

$$\begin{aligned} H : t &= 2iw, \text{ or } \frac{w}{t} = \frac{1}{2i}, \\ H' : t' &= \frac{2}{\sqrt{3}}z'_4 + 3iw' \text{ or } \frac{2}{\sqrt{3}}\frac{z'_4}{t'} + \frac{3iw'}{t'} = 1. \end{aligned}$$

Consider  $\tilde{F} := \rho_5 \circ F \circ \rho_2^{-1} : \mathbb{B}^2 \rightarrow \mathbb{B}^5$  where  $\rho_i$  are the corresponding Cayley transformations. An easy computation shows that the projectivization of  $\tilde{F}$ , denoted by  $\hat{\tilde{F}}$ , is as follows:

$$\begin{aligned} \hat{\tilde{F}}([z : w : t]) &= \left[ z(3t + w) : 2z^2 : 2i\sqrt{\frac{13}{12}}z(t - w) : -\frac{2\sqrt{3}}{3}(t - w)^2 \right. \\ &\quad \left. : \frac{1}{3}(t^2 + 10tw + w^2) : \frac{1}{3}(13t^2 - 2tw + w^2) \right] \end{aligned}$$

and

$$\begin{aligned} \hat{\tilde{H}} &:= \hat{\rho}_2(H) : t = \frac{1}{3}w, \\ \hat{\tilde{H}}' &= \hat{\rho}_5(H') : t' = \frac{\sqrt{3}}{6}z'_4 + \frac{1}{2}w'. \end{aligned}$$

Clearly, we have  $\hat{\tilde{H}} \subset \mathbb{CP}^2$  and  $\hat{\tilde{H}}' \subset \mathbb{CP}^5$  satisfying the property that  $\hat{\tilde{H}} \cap \overline{\mathbb{B}_1^2} = \emptyset$ ,  $\hat{\tilde{H}}' \cap \overline{\mathbb{B}_1^5} = \emptyset$  and

$$\hat{\tilde{F}}(\hat{\tilde{H}}) \subset \hat{\tilde{H}}', \quad \hat{\tilde{F}}(\mathbb{CP}^2 \setminus \hat{\tilde{H}}) \subset \mathbb{CP}^5 \setminus \hat{\tilde{H}}'.$$

According to Lemma 2.1, let

$$\begin{aligned} \hat{\sigma}_1([z : w : t]) &= \left[ \frac{2\sqrt{2}}{3}w : z + \frac{t}{3} : t + \frac{z}{3} \right] \\ \hat{\sigma}_2([z'_1 : z'_2 : z'_3 : z'_4 : w' : t']) &= \left[ \frac{1}{2}(z'_4 + \sqrt{3}w') - \frac{\sqrt{3}}{3}t : \frac{\sqrt{6}}{6}(w' - \sqrt{3}z'_4) \right. \\ &\quad \left. : \frac{\sqrt{6}}{3}z'_1 : \frac{\sqrt{6}}{3}z'_2 : \frac{\sqrt{6}}{3}z'_3 : t - \frac{\sqrt{3}}{6}(z'_4 + \sqrt{3}w') \right], \end{aligned}$$

then  $\hat{\sigma}_1 \in U(3, 1)$  and  $\hat{\sigma}_2 \in U(6, 1)$  with  $\hat{\sigma}_1(\hat{\tilde{H}}_\infty) = \hat{\tilde{H}}$  and  $\hat{\sigma}_2(\hat{\tilde{H}}') = \hat{\tilde{H}}'_\infty$ . The desired proper polynomial holomorphic map  $G$  is thus given by  $\hat{\sigma}_2 \circ \hat{\tilde{F}} \circ \hat{\sigma}_1$ .  $\square$

**Remark 3.4:** It may be interesting to notice that the map  $G$  in Example 3.3 does not preserve the origin and does not equivalent to a map of the form  $(G', 0)$ . We do not know other examples of polynomial proper holomorphic maps between balls of this type.

## 4 Non-polynomially equivalent proper rational holomorphic maps

In this section, we apply Theorem 2.2 to construct examples of rational holomorphic maps which are not equivalent to polynomial holomorphic maps.

**Example 4.1:** Let  $G(z, w) = \left( z^2, \sqrt{2}zw, w^2\left(\frac{z-a}{1-\bar{a}z}, \frac{\sqrt{1-|a|^2}w}{1-\bar{a}z}\right) \right)$ ,  $|a| < 1$ , be a map in  $\text{Rat}(\mathbb{B}^2, \mathbb{B}^4)$ . Then  $G$  is equivalent to a proper polynomial holomorphic map in  $\text{Poly}(\mathbb{B}^2, \mathbb{B}^4)$  if and only if  $a = 0$ .

*Proof:* Indeed, we have

$$\hat{G} = \left[ (t - \bar{a}z)z^2 : (t - \bar{a}z)\sqrt{2}zw : w^2(z - at) : w^2\sqrt{1 - |a|^2}w : (t^3 - \bar{a}t^2z) \right].$$

Suppose there exist hyperplanes  $H = \{\mu_1 z_1 + \mu_2 w + \mu_0 t = 0\} \subset \mathbb{CP}^2$  and  $H' = \{\sum_{j=1}^4 \lambda_j z'_j + \lambda_0 t' = 0\} \subset \mathbb{CP}^4$  such that  $H \cap \overline{\mathbf{S}}_1^2 = \emptyset$ ,  $H' \cap \overline{\mathbf{S}}_1^4 = \emptyset$  and  $\hat{F}(H) \subset H'$ ,  $\hat{F}(\mathbb{CP}^2 \setminus H) \subset \mathbb{CP}^4 \setminus H'$ . Then

$$\begin{aligned} \lambda_1(t - \bar{a}z)z^2 + \lambda_2(t - \bar{a}z)\sqrt{2}zw + \lambda_3w^2(z - at) + \lambda_4w^2\sqrt{1 - |a|^2}w \\ + \lambda_0(t^3 - \bar{a}t^2z) = (\mu_1z + \mu_2w + \mu_0t)^3 \quad \forall [z : w : t] \in \mathbb{CP}^2. \end{aligned}$$

Apparently  $\lambda_0 \neq 0$ . Hence we can assume that  $\lambda_0 = 1, \mu_0 = 1$ . Therefore by comparing the coefficients we get

$$\begin{aligned} \mu_1^3 = -\bar{a}\lambda_1, \quad \mu_2^3 = \lambda_4\sqrt{1 - |a|^2}, \quad 3\mu_2 = 0, \quad 3\mu_1 = 0 \\ 3\mu_1^2 = \lambda_1, \quad 6\mu_1\mu_2 = \sqrt{2}\lambda_2, \quad 3\mu_1^2 = \lambda_3 = 0. \end{aligned}$$

We then have  $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \mu_1 = \mu_2 = 0$ . Moreover, the above equality can only hold when  $a = 0$ . By Theorem 2.2, we see the conclusion.  $\square$

**Example 4.2:** Let  $F(z', w) = \left( z', wz', w^2\left(\frac{\sqrt{1-|a|^2}z'}{1-\bar{a}w}, \frac{w-a}{1-\bar{a}w}\right) \right)$  with  $|a| < 1$  be a map in  $\text{Rat}(\mathbb{B}^n, \mathbb{B}^{3n-2})$ . Then  $F$  has geometric rank 1 and is linear along each hyperplane defined by

$w = \text{constant}$ .  $F$  is equivalent to a proper polynomial holomorphic map in  $\text{Poly}(\mathbb{B}^n, \mathbb{B}^{3n-2})$  if and only if  $a = 0$ .

*Proof:* The projectivization of  $F$  is

$$\hat{F} = [tz'(t - \bar{a}w) : twz' : w^2\sqrt{1 - |a|^2}z' : w^2(w - at) : t^2(t - \bar{a}w)].$$

Assume  $a \neq 0$  and suppose there exist hyperplanes  $H \subset \mathbb{CP}^n$  and  $H' \subset \mathbb{CP}^{3n-2}$  such that  $H \cap \overline{\mathbf{S}_1} = \emptyset$ ,  $H' \cap \overline{\mathbf{S}_1^{3n-2}} = \emptyset$  and  $\hat{F}(H) \subset H'$ ,  $\hat{F}(\mathbb{CP}^n \setminus H) \subset \mathbb{CP}^{3n-2} \setminus H'$ . Then

$$\lambda'_1 tz'(t - \bar{a}w) + \lambda'_2 twz' + \lambda'_3 w^2\sqrt{1 - |a|^2}z' + \lambda_n w^2(w - at) + \lambda_0 t^2(t - \bar{a}w) = (\mu_0 t + \mu' z' + \mu_n w)^3$$

for some  $\lambda'_1, \lambda'_2, \lambda'_3, \mu' \in \mathbb{C}^{n-1}$  and  $\lambda_n, \lambda_0, \mu_0, \mu_n \in \mathbb{C}$ .

Then  $\lambda_0 = \mu_0^3 \neq 0$ . We thus can assume at the beginning that  $\lambda_0 = \mu_0 = 1$ .

Since there are no terms like  $z_j^3 (j < n)$  in the left hand side, we conclude that  $\mu' = 0$ . Thus we get

$$\lambda_n w^2(w - at) + t^2(t - \bar{a}w) = (t + \mu_n w)^3.$$

Therefore  $-\bar{a} = 3\mu_n$ ,  $-\lambda_n a = 3\mu_n^2$ ,  $\lambda_n = \mu_n^3$  or  $\mu_n = -\frac{\bar{a}}{3}$  and  $\mu_n = -\frac{3}{a}$ . This contradicts the assumption that  $0 < |a|^2 < 1$ .  $\square$

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